Exam 3 Review Math 202

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Determinants

1. Let $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$. Compute $det(A)$.

$$det(A) = 2\cdot 4 - 3\cdot 1 = 8-3 = 5$$

2. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$. Compute $det(A)$.

$$det(A) = 1\cdot\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} - 2\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} + 3\begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= 1(1-(-2)) - 2(3-2) + 3(-3-1)$$

$$= 3 - 2 - 12$$

$$= -11$$

3. Let $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 5 \\ 2 & 6 & 0 \end{bmatrix}$. Compute $det(A)$.

$$det(A) = 0\cdot\begin{vmatrix} 0 & 5 \\ 6 & 0 \end{vmatrix} - 1\cdot\begin{vmatrix} 1 & 5 \\ 2 & 0 \end{vmatrix} + 2\begin{vmatrix} 1 & 0 \\ 2 & 6 \end{vmatrix}$$

$$= 0 - 1(0-10) + 2(6-0)$$

$$= 0+10+12$$

$$= 22$$

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4. Let $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 & 5 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$. Compute $det(A)$.

$$det(A) = \underbrace{0 \cdot \begin{vmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix}}_{0} - 1 \cdot \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{vmatrix} + 0 \cdot \underbrace{\begin{vmatrix} 2 & 5 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 3 \end{vmatrix}}_{0} - 0 \cdot \underbrace{\begin{vmatrix} 2 & 5 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{vmatrix}}_{0}$$

$$= -1\left(2\begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} - \underbrace{0\begin{vmatrix} 0 & 0 \\ 1 & 3 \end{vmatrix}}_{0} + 0 \cdot \underbrace{\begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix}}_{0}\right)$$

$$= (-1)(2)(2 \cdot 3 - 0 \cdot 0)$$

$$= -12$$

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5. You can compute the determinant of a large matrix by *reducing to echelon form* using only interchange and replacement, tracking how the determinant changes at each step.

   (a) Interchange: flips the sign of the determinant

$$det\begin{bmatrix} — r_1 — \\ — r_2 — \\ — r_3 — \end{bmatrix} = (-1)\cdot det\begin{bmatrix} — r_2 — \\ — r_1 — \\ — r_3 — \end{bmatrix}$$

   (b) Replacement: doesn't change sign of the Determinant.

$$det\begin{bmatrix} — r_1 — \\ — r_2 — \\ — r_3 — \end{bmatrix} = det\begin{bmatrix} — r_1 — \\ — r_2 + 3r_1 — \\ — r_3 — \end{bmatrix} \quad r_2^* = r_2 + 3r_1$$

   (c) Rescaling: If you can factor a const. out of a row, you can pull it through the determinant

$$det\begin{bmatrix} — r_1 — \\ — 2\cdot r_2 — \\ — r_3 — \end{bmatrix} = (2)\cdot det\begin{bmatrix} — r_1 — \\ — r_2 — \\ — r_3 — \end{bmatrix} \quad r_2^* = \tfrac{1}{2}r_2.$$

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6. Let $A = \begin{bmatrix} 1 & 4 & 2 & 4 \\ 2 & 8 & 4 & 10 \\ 0 & 4 & 6 & 0 \\ 0 & 2 & 2 & 0 \end{bmatrix}$. Compute $det(A)$.

$$\det A = \det\begin{bmatrix} 1 & 4 & 2 & 4 \\ 0 & 0 & 0 & 2 \\ 0 & 4 & 6 & 0 \\ 0 & 2 & 2 & 0 \end{bmatrix} \quad \text{replacement}$$

$$= (-1)\det\begin{bmatrix} 1 & 4 & 2 & 4 \\ 0 & 2 & 2 & 0 \\ 0 & 4 & 6 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad \text{interchange}$$

$$= (-1)\cdot\det\begin{bmatrix} 1 & 4 & 2 & 4 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad \text{replacement}$$

$$= (-1)\cdot 1\cdot 2\cdot 2\cdot 2 \quad \text{determinant of a triangular matrix.}$$

$$= -8$$

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Vector Spaces and Subspaces

1. State the definition of "V is a real *vector space*"

V is a set of elements that can be scaled & added

such that ① V is closed under scaling

② V is closed under adding

③ adding & scaling works like normal

2. Let $\mathbb{P}_2 = \{ \; at^2 + bt + c \; : \; a, b, c \in \mathbb{R} \; \}$. Show that $\mathbb{P}_2$ is closed under scaling and adding.

$$(at^2+bt+c)+(dt^2+et+f) = (a+d)t^2+(b+e)t+(c+f)$$

still in $\mathbb{R}$

$$r(at^2+bt+c) = (ra)t^2+(rb)t+(rc)$$

still in $\mathbb{R}$.

3. Let V be a vector space. State the definition of a "H is a *subspace* of V".

$H \subseteq V$ satisfies

① $\vec{0} \in H$

② H is closed under scaling

③ H is closed under adding

4. Complete the blank in the the following.

Theorem:

$H \subseteq V$ is a subspace of V

and only if

there is a set of vectors $\mathscr{U} \subset H$ so that $H =$ Span $\mathscr{U}$

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5. Let $H = \left\{ \begin{bmatrix} 2t+s \\ t-3s \\ t \end{bmatrix} \in \mathbb{R}^3 : s,t \in R \right\}$ be a subspace of $\mathbb{R}^3$. Prove that H is the span of a set of vectors by *rewriting the set-builder notation.* $\boxed{\text{You must show } \textit{all steps.}}$

$$H = \left\{ \begin{bmatrix} 2t + s \\ t - 3s \\ t + 0s \end{bmatrix} : s,t \in \mathbb{R} \right\}$$

$$= \left\{ t\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + s\begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} : s,t \in \mathbb{R} \right\}$$

$$= \left\{ \vec{b} \in \mathbb{R}^3 : \vec{b} = t\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + s\begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} \text{ for } s,t \in \mathbb{R} \right\}$$

$$= \text{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} \right\}$$

6. Let $H = \left\{ \begin{bmatrix} 3a+2b \\ c \\ a-3b \\ b+2c \end{bmatrix} \in \mathbb{R}^4 : a,b,c \in R \right\}$ be a subspace of $\mathbb{R}^4$. Prove that H is the span of a set of vectors by *rewriting the set-builder notation.* $\boxed{\text{You must show } \textit{all steps.}}$

$$H = \left\{ \begin{bmatrix} 3a + 2b + 0 \\ 0 + 0 + c \\ a - 3b + 0 \\ 0 + b + 2c \end{bmatrix} : a,b,c \in \mathbb{R} \right\}$$

$$= \left\{ a\begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 2 \\ 0 \\ -3 \\ 1 \end{bmatrix} + c\begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} : a,b,c \in \mathbb{R} \right\}$$

$$= \left\{ \vec{b} \in \mathbb{R}^4 : \vec{b} = a\begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 2 \\ 0 \\ -3 \\ 1 \end{bmatrix} + c\begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} \text{ for } a,b,c \in \mathbb{R} \right\}$$

$$= \text{Span}\left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} \right\}$$

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7. Let

$$H = \left\{ \begin{bmatrix} 2p-3q \\ 4p \\ \underline{\underline{1}}+q \end{bmatrix} \in \mathbb{R}^3 : \; p, q \text{ are in } \mathbb{R} \right\}$$

Determine if H is a subspace of $\mathbb{R}^3$. If it is, write it as the span of a set of vectors.

<u>Claim:</u> H is <u>NOT</u> a subspace.

we will prove that $\vec{0} \notin H$.

<u>proof</u> (by computation)

$1+q=0 \Rightarrow q=-1$

$4p=0 \Rightarrow p=0$

<u>BUT</u> $p=0, q=-1 \Rightarrow 2p-3q \neq 0$

<u>SO</u> $\vec{0} \notin H$

<u>SO</u> H is <u>NOT</u> a subspace

<u>ALTERNATE proof</u>

$$H = \left\{ \begin{bmatrix} 2p-3q+0 \\ 4p+0+0 \\ 0+q+1 \end{bmatrix} : p,q \in \mathbb{R} \right\}$$

$$= \left\{ p\begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} + q\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} : p,q \in \mathbb{R} \right\}$$

$$= \left\{ \vec{b} \in \mathbb{R}^3 : \vec{b} = p\begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} + q\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \; p,q \in \mathbb{R} \right\}$$

<u>NOTICE</u>

$\neq$ Span of any ~~vectors~~ set of

H is Not a span

$\Rightarrow H$ is Not a space

8. Let

$$H = \left\{ \begin{bmatrix} 3s+2t \\ t-3s \\ 2t+s \end{bmatrix} \in \mathbb{R}^3 : \; s, t \text{ are in } \mathbb{R} \right\}$$

Determine if H is a subspace of $\mathbb{R}^3$. If it is, write it as the span of a set of vectors.

<u>Claim:</u> H <u>IS</u> a subspace.

we will prove this by writing H as a span.

$$H = \left\{ \begin{bmatrix} 3s+2t \\ -3s+t \\ s+2t \end{bmatrix} : s,t \in \mathbb{R} \right\}$$

$$= \left\{ s\begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix} + t\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} : s,t \in \mathbb{R} \right\}$$

$$= \left\{ \vec{b} \in \mathbb{R}^3 : \vec{b} = s\begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix} + t\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \text{ for } s,t \in \mathbb{R} \right\}$$

$$= \operatorname{Span}\left\{ \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\}.$$

Because H is a span, H is a space

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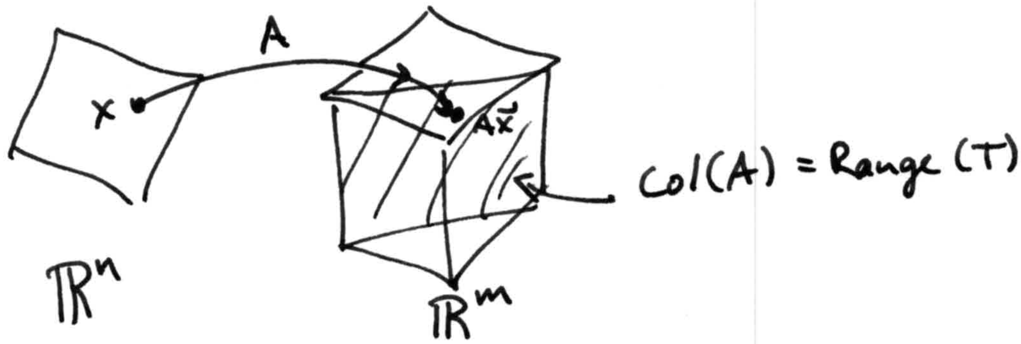
Column Space and Null Space

1. Fix an $m \times n$ matrix $A = \begin{bmatrix} \vec{\mathbf{a}}_1 & \ldots & \vec{\mathbf{a}}_n \end{bmatrix}$. Define the Col(A) and Null(A)

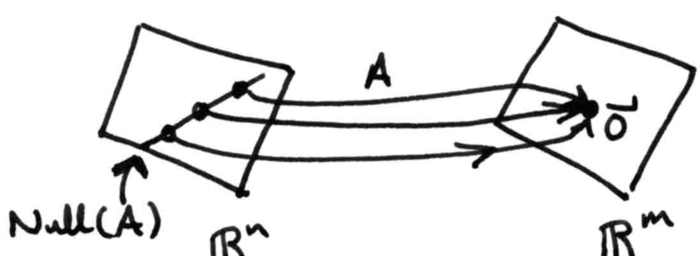
$$\mathrm{Col}(A) = \mathrm{Span}\{\vec{a}_1, \ldots, \vec{a}_n\}$$

$$\mathrm{Null}(A) = \{\vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{0}\}$$

2. Draw a sketch that illustrates Col(A) using the linear transformation $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$



3. Draw a sketch that illustrates Null(A) using the linear transformation $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$



4. What is conveyed by the word "space" in Null*space* and Column *space*?

They are both <u>subspaces</u>.

that is: they contain $\vec{0}$

they are closed under adding

they are closed under scaling

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5. Let $A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 8 & 10 \\ 1 & 5 & 6 \end{bmatrix}$. Determine if $\begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix}$ is in Col(A) and if $\begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}$ is in Null(A) .

$\vec{v} \in Col(A) \iff [A|\vec{v}]$ is consistent

$\iff$

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 2 \\ 2 & 8 & 10 & 6 \\ 1 & 5 & 6 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & 2 \\ 0 & 2 & 2 & 2 \\ 0 & 2 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & 2 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & -2 \end{array}\right]$$

the system is <u>inconsistent</u>

So $\begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix} \notin Col(A)$

$\vec{v} \in Null(A) \iff A\vec{v} = \vec{0}$

<u>Compute</u>:

$$\begin{bmatrix} 1 & 3 & 4 \\ 2 & 8 & 10 \\ 1 & 5 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \checkmark$$

<u>conclude</u>

So $\begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix} \in Null(A)$

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6. Let $A = \begin{bmatrix} -2 & 0 & 4 \\ 0 & 3 & 3 \\ 3 & 2 & -4 \end{bmatrix}$. Find a non-zero vector in Col(A) and a non-zero vector in Null(A).

$$\mathrm{Col}(A) = \mathrm{Span}\left\{ \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ -4 \end{bmatrix} \right\}$$

$$= \left\{ \vec{b} \in \mathbb{R}^m : \vec{b} = c_1 \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 4 \\ 3 \\ -4 \end{bmatrix} \right\}$$

c_1, c_2, c_3 are in $\mathbb{R}$

In particular

picking $c_1 = 0$, $c_2 = 1$, $c_3 = 0$

$$0\begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 4 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} \in \mathrm{Col}(A).$$

$$\mathrm{Null}(A) = \{ \vec{x} \in \mathbb{R}^3 : A\vec{x} = \vec{0} \}$$

Solve: $A\vec{x} = \vec{0}$

$$\left[\begin{array}{ccc|c} -2 & 0 & 4 & 0 \\ 0 & 3 & 3 & 0 \\ 3 & 2 & -4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} +1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 3 & 2 & -4 & 0 \end{array}\right] \begin{array}{l} -\tfrac{1}{2} r_1 \\ \tfrac{1}{3} r_2 \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Leftrightarrow \begin{cases} x_1 - 2x_3 = 0 \\ x_2 + x_3 = 0 \\ x_3 \text{ free} \end{cases} \Leftrightarrow \begin{cases} x_1 = 2x_3 \\ x_2 = -x_3 \\ x_3 \text{ free} \end{cases}$$

Picking $x_3 = 1$ obtain a nonzero vector

$$\vec{x} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

in Null (A).

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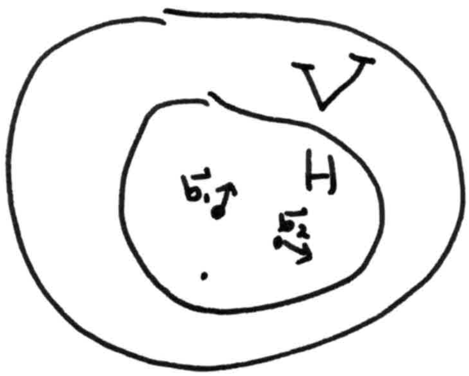
Basis

1. Let H be a subspace of a vector space V, and let $\mathcal{B}$ be a set of vectors in H.

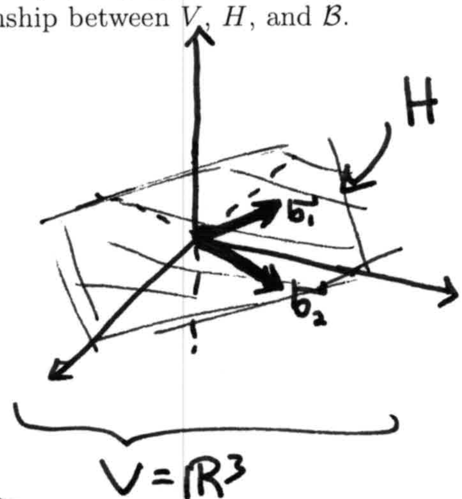
   State the definition of "$\mathcal{B}$ is a basis for H."

$\mathcal{B}$ is a basis for H
if BOTH ① $\mathcal{B}$ is linearly independent
AND ② $H = \text{Span}\, \mathcal{B}$.

2. Suppose that $H \subseteq V$ is a subspace of V, and that $\mathcal{B} = \{b_1, \ldots, b_n\} \subset H$ as in the definition of a basis. Draw a picture that illustrates the relationship between V, H, and $\mathcal{B}$.



OR



3. Let $\mathcal{U} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. Show that $\mathcal{U}$ is a basis for $\mathbb{R}^3$.

   (In other words: check that it satisfies the definition of a basis).

Consider $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

check

A has pivot in ea. column $\Rightarrow$ its columns are independent

A has a pivot in each row $\Rightarrow$ its columns span $\mathbb{R}^3$

Conclude: $\{\text{columns of } A\} = \mathcal{U}$ are a basis for $\mathbb{R}^3$

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4. Determine whether the sets below are bases for $\mathbb{R}^3$. Of the sets that are not bases, determine which ones are linearly independent and which ones span $\mathbb{R}^3$. Justify your answer.

(a) $\mathcal{U} = \left\{ \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 6 \end{bmatrix} \right\}$

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & -2 \\ 4 & 2 & 6 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 6 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

No pivot in col 2 $\Rightarrow$ Not indep

No pivot in row 3 $\Rightarrow$ does not Span $\mathbb{R}^3$

$\mathcal{U}$ is <u>Not</u> a basis for $\mathbb{R}^3$

(b) $\mathcal{U} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 6 \end{bmatrix} \right\}$

$$A = \begin{bmatrix} 1 & 4 & 3 \\ 0 & 3 & 2 \\ 2 & 5 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 3 \\ 0 & 3 & 2 \\ 0 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 3 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

Pivot in all columns $\Rightarrow$ independent

pivot in all rows $\Rightarrow$ spans $\mathbb{R}^3$

$\mathcal{U}$ <u>IS</u> a basis for $\mathbb{R}^3$

(c) $\mathcal{U} = \left\{ \begin{bmatrix} 2 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix} \right\}$

$$A = \begin{bmatrix} 2 & 0 \\ 6 & 4 \\ 0 & 4 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 \\ 0 & 4 \\ 0 & 4 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 \\ 0 & 4 \\ 0 & 0 \end{bmatrix}$$

pivot in each column $\Rightarrow$ <u>IS</u> independent

No pivot in row 3 $\Rightarrow$ does <u>NOT</u> span $\mathbb{R}^3$

$\mathcal{U}$ is <u>NOT</u> a basis for $\mathbb{R}^3$

(d) $\mathcal{U} = \left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \right\}$

$$A = \begin{bmatrix} -1 & 2 & 1 & 2 \\ 0 & 4 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

already in echelon form

<u>NO</u> pivot in column 4 $\Rightarrow$ <u>NOT</u> independent

pivot in each row $\Rightarrow$ <u>does</u> span $\mathbb{R}^3$

$\mathcal{U}$ is <u>NOT</u> a basis for $\mathbb{R}^3$.

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5. Let $\vec{\mathbf{v}}_1 = \begin{bmatrix}1\\0\\3\\0\end{bmatrix}$, $\vec{\mathbf{v}}_2 = \begin{bmatrix}0\\-1\\2\\2\end{bmatrix}$, $\vec{\mathbf{v}}_3 = \begin{bmatrix}2\\-1\\8\\2\end{bmatrix}$, $\vec{\mathbf{v}}_4 = \begin{bmatrix}1\\2\\0\\0\end{bmatrix}$, $\vec{\mathbf{v}}_5 = \begin{bmatrix}2\\0\\7\\4\end{bmatrix}$ and let $H = Span\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3, \vec{\mathbf{v}}_4, \vec{\mathbf{v}}_5\}$.

Is $\mathcal{U} = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3, \vec{\mathbf{v}}_4, \vec{\mathbf{v}}_5\}$ a basis for H? If not, find a basis $\mathcal{B} \subseteq \mathcal{U}$.

NOTICE: more vectors than rows

$\Rightarrow$ $\mathcal{U}$ is NOT independent

(hence not a basis)

$$A = \begin{bmatrix} 1 & 0 & 2 & 1 & 2 \\ 0 & -1 & -1 & 2 & 0 \\ 3 & 2 & 8 & 0 & 7 \\ 0 & 2 & 2 & 0 & 4 \end{bmatrix}$$

$$\sim \begin{array}{cl} \begin{bmatrix} 1 & 0 & 2 & 1 & 2 \\ 0 & -1 & -1 & 2 & 0 \\ 0 & 2 & 2 & -3 & 1 \\ 0 & 2 & 2 & 0 & 4 \end{bmatrix} & \begin{array}{l} \\ \\ r_3 - 3r_1 \\ \\ \end{array} \end{array}$$

$$\sim \begin{array}{cl} \begin{bmatrix} 1 & 0 & 2 & 1 & 2 \\ 0 & -1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 4 & 4 \end{bmatrix} & \begin{array}{l} \\ \\ r_3 + 2r_2 \\ r_4 + 2r_2 \end{array} \end{array}$$

$$\sim \begin{array}{cl} \begin{bmatrix} 1 & 0 & 2 & 1 & 2 \\ 0 & -1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{array}{l} \\ \\ \\ r_4 - 4r_3 \end{array} \end{array}$$

Pivots in rows 1, 2, 4

$\Rightarrow$

a basis for $\mathcal{U}$ is $\mathcal{B} = \left\{ \begin{bmatrix}1\\0\\3\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\2\\2\end{bmatrix}, \begin{bmatrix}1\\2\\0\\0\end{bmatrix} \right\}$

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6. Let $A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 8 & 6 \\ 1 & 5 & 2 \end{bmatrix}$. Find a basis for Col(A) and Null(A).

pivot columns of A form a basis for Col(A)

parametric vector form of $A\vec{x}=\vec{0}$ gives a basis for Null(A)

solve $A\vec{x}=\vec{0}$ to find pivots & solutions

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 2 & 8 & 6 & 0 \\ 1 & 5 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 2 & -2 & 0 \end{array}\right] \begin{array}{l} \\ r_2-2r_1 \\ r_3-r_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 7 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

← need reduced echelon form to find solution set to $A\vec{x}=\vec{0}$

$$\Leftrightarrow \begin{cases} x_1 + 7x_3 = 0 \\ x_2 - x_3 = 0 \\ x_3 \text{ free} \end{cases}$$

$$\vec{x} = \begin{bmatrix} -7x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -7 \\ 1 \\ 1 \end{bmatrix}$$

$\boxed{\mathcal{U} := \left\{ \begin{bmatrix} -7 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ is a basis for Null(A)}}$

Recall: Pivot columns of A give basis for Col(A).

NOTE: pivots in columns 1 & 2 of A

$\Rightarrow$

$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 8 \\ 5 \end{bmatrix} \right\}$ is a basis for Col(A)

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7. Let $A = \begin{bmatrix} -2 & 4 & -2 & 4 & 0 \\ 1 & -2 & -1 & 4 & 1 \\ 0 & 0 & 1 & -3 & 1 \\ 3 & -6 & 0 & 3 & -6 \end{bmatrix}$. You are given the fact that $A \sim B = \begin{bmatrix} 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Find a basis for Col(A) and Null(A).

pivot columns of A form a basis for Col(A)

$$\Rightarrow \mathcal{U} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -6 \end{bmatrix} \right\}$$

is a basis for Col(A)

To find a basis for Null (A) recall that $A \sim B$ implies $[A|\vec{0}] \sim [B|\vec{0}]$

so solutions to $A\vec{x} = \vec{0}$ are given by

$$\left[\begin{array}{ccccc|c} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Leftrightarrow \begin{cases} x_1 - 2x_2 + x_4 = 0 \\ x_2 \text{ free} \\ x_3 - 3x_4 = 0 \\ x_4 \text{ free} \\ x_5 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = 2x_2 - x_4 \\ x_2 \text{ free} \\ x_3 = 3x_4 \\ x_4 \text{ free} \\ x_5 = 0 \end{cases}$$

$$\vec{x} = \begin{bmatrix} 2x_2 - x_4 \\ x_2 + 0 \\ 0 + 3x_4 \\ 0 + x_4 \\ 0 + 0 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix}.$$

$$\text{Null}(A) = \text{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix} \right\}$$

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Define: A set V is a *vector space* if you can *scale* and *add* vectors in V, and

1. V is closed under scaling
2. V is closed under adding
3. scaling & adding works as usual.

Define: Let V be a vector space. A subset $H \subseteq V$ is a *subspace* if

1. $\vec{0} \in H$
2. H is closed under adding
3. H is closed under scaling

Theorem:

$H \subseteq V$ is a subspace of V

and only if

there is a set of vectors $\mathscr{U}$ so that $H =$ Span $\mathscr{U}$

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Fix an $m \times n$ matrix $A = [\vec{a}_1 \dots \vec{a}_n]$

Compare and contrast the Column Space of A and the Null Space of A:

Col(A)	Null(A)
Col(A) $\subseteq \mathbb{R}^m$	Null(A) $\subseteq \mathbb{R}^n$
Col(A) = $\text{Span}\{\vec{a}_1, \dots, \vec{a}_n\}$	Null(A) = $\{\vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{0}\}$
explicitly defined	implicitly defined
easy to find elements : pick scalars.	*hard* to find elements : solve $A\vec{x} = \vec{0}$ (reduce $[A \mid \vec{0}]$)
hard to check if $\vec{\mathbf{v}}$ is an element reduce $[A \mid \vec{v}]$	*easy* check if $\vec{\mathbf{v}}$ is an element compute $A\vec{x}$. check if it $= 0$.
Null(A) $= \{\vec{0}\}$ if and only if $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$ is one-to-one	Col(A) $= \mathbb{R}^m$ if and only if $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$ is onto.